DAY TWELVE

Differentiation

Learning & Revision for the Day

- Derivative (Differential Coefficient)
- Geometrical Meaning of Derivative at a point
- Methods of Differentiation
- Second Order Derivative
- Differentiation of a Determinant

Derivative (Differential Coefficient)

The rate of change of a quantity y with respect to another quantity x is called the **derivative or differential coefficient** of y with respect to x. The process of finding derivative of a function called **differentiation**.

Geometrical Meaning of Derivative at a Point

Geometrically derivative of a function at a point $x = c$ is the slope of the tangent to the curve $y = f(x)$ at the point $P\{c, f(c)\}$.

Slope of tangent at $P = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \left\{\frac{df(x)}{dx}\right\}_{x=c}$ or $f'(c)$.

Derivative of Some Standard Functions

- $\frac{d}{dx}(\text{constant}) = 0$
- $\frac{d}{dx} x^n = nx^{n-1}$
- $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
- $\frac{d}{dx}\left(\frac{1}{x^n}\right) = -\frac{n}{x^{n+1}}$
- $\frac{d}{dx}(\sin x) = \cos x$
- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(\tan x) = \sec^2 x$
- $\frac{d}{dx}(\sec x) = \sec x \tan x$
- $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$
- $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$
- $\frac{d}{dx}(\log x) = \frac{1}{x}$, for $x > 0$
- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(a^x) = a^x \log a$, for $a > 0$
- $\frac{d}{dx}(\log_a x) = \frac{1}{x \log a}$, for $x > 0, a > 0, a \neq 1$

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$
- $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$
- $\frac{d}{dx}(\text{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, for $x \in R$
- $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$, for $x \in R$

Methods of Differentiation

(i) If $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = \frac{d}{dx}\{f(x) \pm g(x)\}$

$= f'(x) \pm g'(x)$

(ii) If $y = c \cdot f(x)$, where c is any constant, then

$\frac{dy}{dx} = \frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x)$. [Scalar multiple rule]

(iii) If $y = f(x) \cdot g(x)$, then $\frac{dy}{dx} = \frac{d}{dx}\{f(x) \cdot g(x)\}$

$= f(x) \cdot g'(x) + g(x) \cdot f'(x)$ [Product rule]

(iv) If $y = \frac{f(x)}{g(x)}$, then $\frac{dy}{dx} = \frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{\{g(x)\}^2}$

$g(x) \neq 0$

(v) If $y = f(u)$ and $u = g(x)$, then

$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x)$ [Chain rule]

This rule can be extended as follows. If $y = f(u), u = g(v)$ and $v = h(x)$, then $\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$,

(vi) $\frac{d}{dx}(f\{g(x)\}) = f'(g(x)) \cdot g'(x)$

(vii) If given function cannot be expressed in the form $y = f(x)$ but can be expressed in the form $f(x, y) = 0$, then to find derivatives of each term of $f(x, y) = 0$ w.r.t x. [differentiation of implicit function]

(viii) If y is the product or the quotient of a number of complicated functions or if it is of the form $(f(x))^{g(x)}$, then the derivative of y can be found by first taking log on both sides and then differentiating it. [logarithmic differentiation rule]

When $y = (f(x))^{g(x)}$, then $\frac{dy}{dx} = (f(x))^{g(x)}$

$\left[\frac{g(x)}{f(x)} \cdot f'(x) + \log f(x) \cdot g'(x)\right]$

(ix) If $x = \phi(t)$ and $y = \Psi(t)$, where t is parameter, then

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ [Parametric differentiation rule]

(x) If $u = f(x)$ and $v = g(x)$, then the differentiation of u with respect to v is $\frac{du}{dv} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}$.

[Differentiation of a function w.r.t another function]

Differentiation Using Substitution

In order to find differential coefficients of complicated expressions, some substitution are very helpful, which are listed below

S. No.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a \sin\theta$ or $a \cos\theta$
(ii)	$\sqrt{x^2 - a^2}$	$x = a \sec\theta$ or $a\,\text{cosec}\,\theta$
(iii)	$\sqrt{x^2 + a^2}$	$x = a \tan\theta$ or $a \cot\theta$
(iv)	$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a \cos 2\theta$
(v)	$\sqrt{\frac{a^2+x^2}{a^2-x^2}}$ or $\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2 = a^2 \cos 2\theta$
(vi)	$\sqrt{\frac{x}{a+x}}$	$x = a \tan^2\theta$
(vii)	$\sqrt{(x-a)(x-b)}$	$x = a \sec^2\theta - b \tan^2\theta$
(viii)	$\sqrt{ax - x^2}$	$x = a \sin^2\theta$
(ix)	$\sqrt{\frac{x}{a-x}}$	$x = a \sin^2\theta$
(x)	$\sqrt{(x-a)(b-x)}$	$x = a \cos^2\theta + b \sin^2\theta$

Usually this is done in case of inverse trigonometric functions.

Second Order Derivative

If $y = f(x)$, then $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the **second order derivative** of y w.r.t x. It is denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$ or y'' or y_2.

Differentiation of a Determinant

If $y = \begin{vmatrix} p & q & r \\ u & v & w \\ l & m & n \end{vmatrix}$, then $\frac{dy}{dx} = \begin{vmatrix} \frac{dp}{dx} & \frac{dq}{dx} & \frac{dr}{dx} \\ u & v & w \\ l & m & n \end{vmatrix}$

$+ \begin{vmatrix} p & q & r \\ \frac{du}{dx} & \frac{dv}{dx} & \frac{dw}{dx} \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ u & v & w \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \end{vmatrix}$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 If $f(x) = |\cos x|$, then $f'\left(\frac{3\pi}{4}\right)$ is equal to → **NCERT Exemplar**

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) $\frac{1}{2}$ (d) $2\sqrt{2}$

2 If $f(x) = |x-1|$ and $g(x) = f[f\{f(x)\}]$, then for $x > 2$, $g'(x)$ is equal to

(a) -1, if $2 \le x < 3$ (b) 1, if $2 \le x < 3$
(c) 1, if $x > 2$ (d) None of these

3 The derivative of $y = (1-x)(2-x)\ldots(n-x)$ at $x = 1$ is

(a) 0 (b) $(-1)(n-1)!$ (c) $n! - 1$ (d) $(-1)^{n-1}(n-1)!$

4 If $f(x) = x^n$, then the value of

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots + \frac{(-1)^n f^n(1)}{n!} \text{ is}$$

(a) 2^n (b) 0 (c) 2^{n-1} (d) None of these

5 If $f(x) = \frac{x^2 - x}{x^2 + 2x}$, where $x \ne 0, -2$, then $\frac{d}{dx}[f^{-1}(x)]$ (whenever it is defined) is equal to → **JEE Mains 2013**

(a) $\frac{-1}{(1-x)^2}$ (b) $\frac{3}{(1-x)^2}$ (c) $\frac{1}{(1-x)^2}$ (d) $\frac{-3}{(1-x)^2}$

6 If $f(x) = 2 + |x| - |x-1| - |x+1|$, then

$f'\left(-\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) + f'\left(\frac{3}{2}\right) + f'\left(\frac{5}{2}\right)$ is equal to

(a) 1 (b) -1 (c) 2 (d) -2

7 If $f(x) = |\log_e |x||$, then $f'(x)$ equals

(a) $\frac{1}{|x|}$, where $x \ne 0$ (b) $\frac{1}{x}$ for $|x| > 1$
(c) $-\frac{1}{x}$ for $|x| > 1$
(d) $\frac{1}{x}$ for $x > 0$ and $-\frac{1}{x}$ for $x < 0$

8 If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

(a) 1 (b) -1 (c) 0 (d) None of these

9 If $f(x) = \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x$, then $f'\left(\frac{\pi}{4}\right)$ is equal to

(a) $\sqrt{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) 1 (d) $\frac{\sqrt{3}}{2}$

10 If $\sin y = x \sin(a+y)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\sin a}{\sin^2(a+y)}$ (b) $\frac{\sin^2(a+y)}{\sin a}$
(c) $\sin a \sin^2(a+y)$ (d) $\frac{\sin^2(a-y)}{\sin a}$

11 If $y = (1-x)(1+x^2)(1+x^4)\ldots(1+x^{2n})$, then $\frac{dy}{dx}$ at $x = 0$ is equal to

(a) -1 (b) $\frac{1}{(1+x)^2}$ (c) $\frac{x}{(1+x^2)}$ (d) $\frac{x}{(1-x)^2}$

12 If $f : (-1, 1) \to R$ be a differentiable function with $f(0) = 1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x)+2)]^2$. The, $g'(0)$ is equal to

(a) 4 (b) -4 (c) 0 (d) -2

13 Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a, b, c are in AP, then $f'(a), f'(b)$ and $f'(c)$ are in.

(a) AP
(b) GP
(c) Arithmetic-Geometric progression
(d) None of the above

14 If $y = f(x)$ is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3) = -2$, then $f'(-3)$ is equal to

(a) 4 (b) 2 (c) -2 (d) 0

15 If f and g are differentiable function satisfying $g'(a) = 2, g(a) = b$ and $fog = I$ (identity function). Then, $f'(b)$ is equal to

(a) $\frac{1}{2}$ (b) 2 (c) $\frac{2}{3}$ (d) None of these

16 If y is an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then, $y'(1)$ is equal to

(a) -1 (b) 1 (c) $\log 2$ (d) $-\log 2$

17 If $x^m y^n = (x+y)^{m+n}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{x+y}{xy}$ (b) xy (c) $\frac{x}{y}$ (d) $\frac{y}{x}$

18 If $y = (x + \sqrt{1+x^2})^n$, then $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx}$ is equal to

(a) $n^2 y$ (b) $-n^2 y$ (c) $-y$ (d) $2x^2 y$

19 If $x = \frac{2t}{1+t^2}$ and $y = \frac{1-t^2}{1+t^2}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{2t}{t^2+1}$ (b) $\frac{2t}{t^2-1}$
(c) $\frac{2t}{1-t^2}$ (d) None of these

20 For $a > 0, t \in \left(0, \frac{\pi}{2}\right)$, let $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$.

Then, $1 + \left(\frac{dy}{dx}\right)^2$ equals → **JEE Mains 2013**

(a) $\frac{x^2}{y^2}$ (b) $\frac{y^2}{x^2}$ (c) $\frac{x^2+y^2}{y^2}$ (d) $\frac{x^2+y^2}{x^2}$

21 If $y = \sec^{-1}\left[\frac{\sqrt{x}+1}{\sqrt{x}-1}\right] + \sin^{-1}\left[\frac{\sqrt{x}-1}{\sqrt{x}+1}\right]$, then $\frac{dy}{dx}$ is equal to

(a) 0 (b) $\frac{1}{\sqrt{x}+1}$

(c) 1 (d) None of these

22 For $x \in \left(0, \frac{1}{4}\right)$, if the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x} \cdot g(x)$, then $g(x)$ equals → **JEE Mains 2017**

(a) $\frac{9}{1+9x^3}$ (b) $\frac{3x\sqrt{x}}{1-9x^3}$ (c) $\frac{3x}{1-9x^3}$ (d) $\frac{3}{1+9x^3}$

23 If $y = \sec(\tan^{-1} x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to → **JEE Mains 2013**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) 1 (d) $\sqrt{2}$

24 If $y = \tan^{-1}\left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right)$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{x}{\sqrt{1-x^4}}$ (b) $-\frac{x}{\sqrt{1-x^2}}$ (c) $\frac{2x}{\sqrt{1-x^4}}$ (d) None of these

25 If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, then $\frac{dy}{dx}$ is equal to

(a) $\sqrt{\frac{1-x^2}{1-y^2}}$ (b) $\sqrt{\frac{1-y^2}{1-x^2}}$ (c) $\sqrt{\frac{x^2-1}{1-y^2}}$ (d) $\sqrt{\frac{y^2-1}{1-x^2}}$

26 If $y = \sin^{-1}(x\sqrt{1-x} + \sqrt{x}\sqrt{1-x^2})$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{-2x}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{x-x^2}}$ (b) $\frac{-1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x-x^2}}$

(c) $\frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{x-x^2}}$ (d) None of these

27 If $y = \frac{a^{\cos^{-1} x}}{1+a^{\cos^{-1} x}}$ and $z = a^{\cos^{-1} x}$, then $\frac{dy}{dz}$ is equal to

(a) $-\frac{1}{1+a^{\cos^{-1} x}}$ (b) $\frac{1}{1+a^{\cos^{-1} x}}$

(c) $\frac{1}{(1+a^{\cos^{-1} x})^2}$ (d) None of these

28 Let $g(x)$ be the inverse of $f(x)$ such that $f'(x) = \frac{1}{1+x^5}$, then $\frac{d^2(g(x))}{dx^2}$ is equal to

(a) $\frac{1}{1+(g(x))^5}$ (b) $\frac{g'(x)}{1+(g(x))^5}$

(c) $5(g(x))^4(1+(g(x))^5$ (d) $1+(g(x))^5$

29 $\frac{d^2x}{dy^2}$ is equal to → **AIEEE 2011**

(a) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (b) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$

(c) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (d) $\left(\frac{d^2y}{dx^2}\right)^{-1}$

30 If $y = \tan^{-1}\left(\frac{\log(e/x^2)}{\log(ex^2)}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$, then $\frac{d^2y}{dx^2}$ is

(a) 2 (b) 1
(c) 0 (d) −1

31 If graph of $y = f(x)$ is symmetrical about the Y-axis and that of $y = g(x)$ is symmetrical about the origin and if $h(x) = f(x) \cdot g(x)$, then $\frac{d^2h(x)}{dx^2}$ at $x = 0$ is

(a) $f(0)g(0)$
(b) 0
(c) can't be determined
(d) None of the above

32 If $\begin{vmatrix} f'(x) & f(x) \\ f''(x) & f'(x) \end{vmatrix} = 0$, where $f(x)$ is continuously differentiable function with $f'(x) \neq 0$ and satisfies $f(0) = 1$ and $f'(0) = 2$, then $\lim_{x \to 0} \frac{f(x)-1}{x}$ is

(a) 1 (b) 2
(c) $\frac{1}{2}$ (d) 0

33 If $f(x) = \begin{vmatrix} 3x^2 & \cos x & \sin x \\ 6 & -1 & 0 \\ P & P^2 & P^3 \end{vmatrix}$, where P is a constant.

Then, $\frac{d^2}{dx^2}\{f(x)\}$ at $x = 0$ is equal to

(a) P (b) $P + P^2$
(c) $P + P^3$ (d) independent of P

34 Which of the following statements is/are true?

Statement I If $y = (\log x)^{\log x}$, then
$$\frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right].$$

Statement II If $y = \cos(a\cos x + b\sin x)$ for some constants a and b, then
$$y' = (a\sin x - b\cos x)\sin(a\cos x + b\sin x)$$

(a) Only I is true
(b) Only II is true
(c) Both I and II are true
(d) Neither I nor II is true

35 **Statement I** If $u = f(\tan x)$, $v = g(\sec x)$ and $f'(1) = 2$, $g'(\sqrt{2}) = 4$, then $\left(\frac{du}{dv}\right)_{x=\pi/4} = \frac{1}{\sqrt{2}}$.

Statement II If $u = f(x)$, $v = g(x)$, then the derivative of f with respect to g is $\frac{du}{dv} = \frac{du/dx}{dv/dx}$.

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

1 For $x \in R, f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then

(a) g is not differentiable at $x = 0$
(b) $g'(0) = \cos(\log 2)$
(c) $g'(0) = -\cos(\log 2)$
(d) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

2 If $y = \sin x \cdot \sin 2x \cdot \sin 3x \ldots \sin nx$, then y' is

(a) $\sum_{k=1}^{n} k \cdot \tan kx$ (b) $y \cdot \sum_{k=1}^{n} k \cot kx$
(c) $y \cdot \sum_{k=1}^{n} k \cdot \tan kx$ (d) $\sum_{k=1}^{n} \cot kx$

3 If $3f(x) - 2f(1/x) = x$, then $f'(2)$ is equal to

(a) $\frac{2}{7}$ (b) $\frac{1}{2}$ (c) 2 (d) $\frac{7}{2}$

4 If $f(x) = (\cos x + i \sin x) \cdot (\cos 2x + i \sin 2x)$ $(\cos 3x + i \sin 3x) \ldots (\cos nx + i \sin nx)$ and $f(1) = 1$, then $f''(1)$ is equal to

(a) $\frac{n(n+1)}{2}$ (b) $\left[\frac{n(n+1)}{2}\right]^2$
(c) $-\left[\frac{n(n+1)}{2}\right]^2$ (d) None of these

5 If $\sqrt{x^2 + y^2} = ae^{\tan^{-1}\left(\frac{y}{x}\right)}$, $a>0$ assuming $y > 0$, then $y''(0)$ is equal to

(a) $\frac{2}{a}e^{-\pi/2}$ (b) $-\frac{2}{a}e^{\pi/2}$
(c) $-\frac{2}{a}e^{-\pi/2}$ (d) None of these

6 If $y = |\sin x|^{|x|}$, then the value of $\frac{dy}{dx}$ at $x = -\frac{\pi}{6}$ is

(a) $\frac{2^{\frac{-\pi}{6}}}{6}[6\log 2 - \sqrt{3}\pi]$ (b) $\frac{2^{\frac{\pi}{6}}}{6}[6\log 2 + \sqrt{3}\pi]$
(c) $\frac{2^{-\frac{\pi}{6}}}{6}[6\log 2 + \sqrt{3}\pi]$ (d) None of these

7 The solution set of $f'(x) > g'(x)$, where $f(x) = \frac{1}{2}(5)^{2x+1}$ and $g(x) = 5^x + 4x \log_e 5$ is

(a) $(1, \infty)$ (b) $(0, 1)$ (c) $(\infty, 0)$ (d) $(0, \infty)$

8 Let $f''(x) = -f(x)$, where $f(x)$ is a continuous double differentiable function and $g(x) = f'(x)$.

If $F(x) = \left[f\left(\frac{x}{2}\right)\right]^2 + \left[g\left(\frac{x}{2}\right)\right]^2$ and $F(5) = 5$, then $F(10)$ is equal to

(a) 0 (b) 5
(c) 10 (d) 25

9 If $f(2) = 4, f'(2) = 3, f''(2) = 1$, then $(f^{-1})''(4)$ is equal to

(a) $\frac{-1}{9}$ (b) $\frac{-1}{81}$
(c) $\frac{-1}{27}$ (d) $\frac{-1}{3}$

10 If $f(x) = \sin(\sin x)$ and $f''(x) + \tan x\, f'(x) + g(x) = 0$, then $g(x)$ is equal to **➜ JEE Mains 2013**

(a) $\cos^2 x \cos(\sin x)$ (b) $\sin^2 x \cos(\cos x)$
(c) $\sin^2 x \sin(\cos x)$ (d) $\cos^2 x \sin(\sin x)$

11 If $x = a \cos t \sqrt{\cos 2t}$ and $y = a \sin t \sqrt{\cos 2t}$ (where, $a > 0$), then $\left|\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\right|$ at $\frac{\pi}{6}$ is given by

(a) $\frac{a}{3}$ (b) $a\sqrt{2}$
(c) $\frac{\sqrt{2}}{3a}$ (d) $\frac{\sqrt{2}a}{3}$

12 Let $f(x) = e^{\ln e^{g(x)}}$ and $g(x+1) = x + g(x)\ \forall\ x \in R$. If $n \in I^+$, then $\frac{f'\left(n+\frac{1}{3}\right)}{f\left(n+\frac{1}{3}\right)} - \frac{f'\left(\frac{1}{3}\right)}{f\left(\frac{1}{3}\right)}$ is equal to

(a) $3\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$ (b) $3\left(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2n-1}\right)$
(c) n (d) 1

13 If the function $f(x) = -4e^{\frac{1-x}{2}} + 1 + x + \frac{x^2}{2} + \frac{x^3}{3}$ and $g(x) = f^{-1}(x)$, then the value of $g'\left(\frac{-7}{6}\right)$ is equal to

(a) $\frac{1}{5}$ (b) $-\frac{1}{5}$
(c) $\frac{6}{7}$ (d) $-\frac{6}{7}$

14. If $f(x) = (x-1)^{100}(x-2)^{2(99)}(x-3)^{3(98)} \ldots (x-100)^{100}$, then the value of $\frac{f'(101)}{f(101)}$ is

(a) 5050 (b) 2575
(c) 3030 (d) 1250

15 The derivative of the function represented parametrically as $x = 2t - |t|,\ y = t^3 + t^2|t|$ at $t = 0$ is

(a) -1 (b) 0
(c) 1 (d) does not exist.

ANSWERS

SESSION 1	**1** (a)	**2** (a)	**3** (b)	**4** (b)	**5** (b)	**6** (d)	**7** (b)	**8** (a)	**9** (a)	**10** (b)
	11 (a)	**12** (b)	**13** (a)	**14** (c)	**15** (a)	**16** (a)	**17** (d)	**18** (a)	**19** (b)	**20** (d)
	21 (a)	**22** (a)	**23** (a)	**24** (a)	**25** (b)	**26** (c)	**27** (c)	**28** (c)	**29** (c)	**30** (c)
	31 (b)	**32** (b)	**33** (d)	**34** (c)	**35** (a)					
SESSION 2	**1** (b)	**2** (b)	**3** (b)	**4** (c)	**5** (c)	**6** (a)	**7** (d)	**8** (b)	**9** (c)	**10** (d)
	11 (d)	**12** (c)	**13** (a)	**14** (a)	**15** (b)					

Hints and Explanations

SESSION 1

1 When $\frac{\pi}{2} < x < \pi$, $\cos x < 0$, so that

$|\cos x| = -\cos x$,

i.e. $f(x) = -\cos x$, $f'(x) = \sin x$

Hence, $f'\left(\frac{3\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$

2 We have, $f(x) = |x-1|$ $\quad [\because x > 2]$

$$f[f(x)] = f(x-1) = |(x-1)-1| = |x-2|$$

$$g(x) = f[f\{f(x)\}] = f(x-2) = |(x-2)-1| = |x-3| = \begin{cases} x-3, & \text{if } x \geq 3 \\ -x+3, & \text{if } 2 \leq x < 3 \end{cases}$$

$$\therefore g'(x) = \begin{cases} 1, & \text{if } x \geq 3 \\ -1, & \text{if } 2 \leq x < 3 \end{cases}$$

3 $$\frac{dy}{dx} = -[(2-x)(3-x)\ldots(n-x) + (1-x)(3-x)\ldots(n-x) + \ldots + (1-x)(2-x)\ldots(n-1-x)]$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = -[(n-1)! + 0 + \ldots + 0] = (-1)(n-1)!$$

4 We have, $f(x) = x^n$

$$\Rightarrow f(1) = 1 = {}^nC_0$$

$$\frac{f'(1)}{1!} = \frac{n}{1!} = {}^nC_1$$

$$\Rightarrow \frac{f''(1)}{2!} = \frac{n(n-1)}{2!} = {}^nC_2$$

$$\frac{f'''(1)}{3!} = \frac{n(n-1)(n-2)}{3!} = {}^nC_3$$

$$\vdots$$

$$\frac{f^n(1)}{n!} = \frac{n!}{n!} = {}^nC_n$$

$$\therefore f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \ldots + \frac{(-1)^n f^n(1)}{n!}$$

$$= {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n\, {}^nC_n$$

$$= (1-1)^n = 0$$

5 Let $y = \frac{x^2 - x}{x^2 + 2x}$

$$\Rightarrow x = \frac{2y+1}{-y+1}; x \neq 0$$

$$\Rightarrow f^{-1}(x) = \frac{2x+1}{-x+1}$$

$$\therefore \frac{d}{dx}[f^{-1}(x)] = \frac{(-x+1)\cdot 2 - (2x+1)(-1)}{(-x+1)^2} = \frac{3}{(-x+1)^2}$$

6 We have, $f(x) = 2 + |x| - |x-1| - |x+1|$

$$\therefore f(x) = \begin{cases} 2-x+(x-1)+(x+1), & \text{if } x < -1 \\ 2-x+(x-1)-(x+1), & \text{if } -1 \leq x < 0 \\ 2+x+(x-1)-(x+1), & \text{if } 0 \leq x < 1 \\ 2+x-(x-1)-(x+1), & \text{if } x \geq 1 \end{cases}$$

$$= \begin{cases} x+2, & \text{if } x < -1 \\ -x, & \text{if } -1 \leq x < 0 \\ x, & \text{if } 0 \leq x < 1 \\ 2-x, & \text{if } x \geq 1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 1, & \text{if } x < -1 \\ -1, & \text{if } -1 \leq x < 0 \\ 1, & \text{if } 0 \leq x < 1 \\ -1, & \text{if } x \geq 1 \end{cases}$$

Hence, $f'\left(-\frac{1}{2}\right) + f'\left(\frac{1}{2}\right) + f'\left(\frac{3}{2}\right) + f'\left(\frac{5}{2}\right) = (-1) + 1 + (-1) + (-1) = -2$

7 We have, $f(x) = |\log_e |x||$

$$\therefore f(x) = \begin{cases} \log(-x), & x < -1 \\ -\log(-x), & -1 < x < 0 \\ -\log x, & 0 < x < 1 \\ \log x, & x > 1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} \frac{1}{x}, & x < -1 \\ -\frac{1}{x}, & -1 < x < 0 \\ -\frac{1}{x}, & 0 < x < 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

Clearly, $f'(x) = \frac{1}{x}$ for $|x| > 1$

8 When $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$

$\therefore \cos x - \sin x > 0$

Also, when $\frac{\pi}{4} < x < \pi$, $\cos x < \sin x$

$\therefore \cos x - \sin x < 0$

$\therefore |\cos x - \sin x| = -(\cos x - \sin x)$, when $\frac{\pi}{4} < x < \pi$

$\Rightarrow f'(x) = \sin x + \cos x$

$\Rightarrow f'\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \cos\frac{\pi}{2} = 1 + 0 = 1$

9 $$f(x) = \frac{2\sin x \cdot \cos x \cdot \cos 2x \cdot \cos 4x \cdot \cos 8x \cdot \cos 16x}{2\sin x}$$

$$= \frac{\sin 2x \cos 2x \cos 4x \cdot \cos 8x \cdot \cos 16x}{2 \sin x}$$

$$= \frac{\sin 32x}{2^5 \sin x}$$

$\therefore\ f'(x) = \frac{1}{32} \cdot \frac{32\cos 32x \cdot \sin x - \cos x \cdot \sin 32x}{\sin^2 x}$

$\Rightarrow f'\left(\frac{\pi}{4}\right) = \frac{32 \times \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \times 0}{32\left[\frac{1}{\sqrt{2}}\right]^2} = \sqrt{2}$

10 $\therefore\ \sin y = x\sin(a+y)$

$\Rightarrow\ x = \frac{\sin y}{\sin(a+y)}$

On differentiating w.r.t. y, we get

$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}$

$\Rightarrow \frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$

$\Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

11 Given, $y = (1-x)(1+x^2)(1+x^4)\ldots(1+x^{2n})$

or $y = \frac{(1-x^2)(1+x^2)\ldots(1+x^{2n})}{(1+x)}$

$= \frac{1-(x)^{4n}}{(1+x)}$

$\therefore \frac{dy}{dx} = \frac{(1+x)\cdot(0-4n\cdot x^{4n-1}) - (1-x^{4n})\cdot 1}{(1+x)^2}$

$\therefore \left(\frac{dy}{dx}\right)_{x=0} = -1$

12 We have, $f:(-1,1) \to R$

$f(0) = -1,\ f'(0) = 1$

$g(x) = [f(2f(x)+2)]^2$

$\Rightarrow g'(x) = 2[f(2f(x)+2)] \times f'(2f(x)+2) \times 2f'(x)$

$\Rightarrow g'(0) = 2[f\{2f(0)+2\}] \times f'\{2f(0)+2\} \times 2f'(0)$

$= 2[f(0)] \times f'(0) \times 2f'(0)$

$= 2 \times (-1) \times 1 \times 2 \times 1 = -4$

13 Let $f(x) = Ax^2 + Bx + C$

$\therefore\ f(1) = A + B + C$

and $f(-1) = A - B + C$

$\because\ f(1) = f(-1)$ [given]

$\Rightarrow A + B + C = A - B + C$

$\Rightarrow\ 2B = 0 \Rightarrow B = 0$

$\therefore\ f(x) = Ax^2 + C$

$\Rightarrow f'(x) = 2Ax$

$\therefore\ f'(a) = 2Aa$

$f'(b) = 2Ab$ and $f'(c) = 2Ac$

Also, a, b, c are in AP.

So, $2Aa, 2Ab$ and $2Ac$ are in AP.

Hence, $f'(a), f'(b)$ and $f'(c)$ are also in AP.

14 Since, $f(x)$ is odd.

$\therefore\ f(-x) = -f(x)$

$\Rightarrow\ f'(-x)(-1) = -f'(x)$

$\Rightarrow\ f'(-x) = f'(x)$ $\therefore$

$f'(-3) = f'(3) = -2$

15 Since, $fog = I \Rightarrow fog(x) = x$ for all x

$\Rightarrow\ f'(g(x))g'(x) = 1$ for all x

$\Rightarrow\ f'(g(a)) = \frac{1}{g'(a)} = \frac{1}{2}$

$\Rightarrow\ f'(b) = \frac{1}{2}$ $\quad [\because g(a) = b]$

16 $x^{2x} - 2x^x \cot y - 1 = 0$ $\quad$ ...(i)

Now, $x = 1$,

$1 - 2\cot y - 1 = 0$

$\Rightarrow\ \cot y = 0$

$\Rightarrow\ y = \frac{\pi}{2}$

On differentiating Eq. (i) w.r.t. x, we get

$2x^{2x}(1+\log x) - 2\left[x^x(-\operatorname{cosec}^2 y)\frac{dy}{dx} + \cot y\ x^x(1+\log x)\right] = 0$

At $\left(1, \frac{\pi}{2}\right)$, $2(1+\log 1) - 2\left\{1(-1)\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} + 0\right\} = 0$

$\Rightarrow 2 + 2\left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = 0$

$\therefore\ \left(\frac{dy}{dx}\right)_{\left(1,\frac{\pi}{2}\right)} = -1$

17 Given that, $x^m y^n = (x+y)^{m+n}$

Taking log on both sides, we get

$m\log x + n\log y = (m+n)\log(x+y)$

On differentiating w.r.t. x, we get

$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \frac{(m+n)}{(x+y)}\left(1 + \frac{dy}{dx}\right)$

$\Rightarrow \frac{dy}{dx}\left(\frac{m+n}{x+y} - \frac{n}{y}\right) = \frac{m}{x} - \frac{m+n}{x+y}$

$\Rightarrow \frac{dy}{dx}\left(\frac{my + ny - nx - ny}{y(x+y)}\right) = \frac{mx + my - mx - nx}{x(x+y)}$

$\therefore\ \frac{dy}{dx} = \frac{y}{x}$

18 $\frac{d}{dx}(y) = n(x + \sqrt{1+x^2})^{n-1}\left(1 + \frac{x}{\sqrt{1+x^2}}\right)$

$\Rightarrow (\sqrt{1+x^2})\frac{dy}{dx} = n(x + \sqrt{1+x^2})^n$

$\Rightarrow \frac{d^2y}{dx^2}(\sqrt{1+x^2}) + \frac{dy}{dx}\left(\frac{x}{\sqrt{1+x^2}}\right) = n^2(x + \sqrt{1+x^2})^{n-1}\left(1 + \frac{x}{\sqrt{1+x^2}}\right)$

$\Rightarrow \frac{\frac{d^2y}{dx^2}(1+x^2) + \frac{dy}{dx}\cdot x}{\sqrt{1+x^2}} = \frac{n^2(x + \sqrt{1+x^2})^n}{\sqrt{1+x^2}}$

$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2(x + \sqrt{1+x^2})^n$

$\Rightarrow (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2 y$

19 We have, $x = \frac{2t}{1+t^2}, y = \frac{1-t^2}{1+t^2}$

Put $t = \tan\theta$

$\therefore\ x = \frac{2\tan\theta}{1+\tan^2\theta} = \sin 2\theta$ and

$y = \frac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$

$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2\sin 2\theta}{2\cos 2\theta} = -\tan 2\theta$

$= \frac{-2\tan\theta}{1-\tan^2\theta}$

$= \frac{-2t}{1-t^2} = \frac{2t}{t^2-1}$

20 $\because \frac{dx}{dt} = \frac{1}{2\sqrt{a^{\sin^{-1}t}}}\left(a^{\sin^{-1}t} \times \frac{1}{\sqrt{1-t^2}}\right)$

and $\frac{dy}{dt} = \frac{1}{2\sqrt{a^{\cos^{-1}t}}}\left(a^{\cos^{-1}t} \times \frac{-1}{\sqrt{1-t^2}}\right)$

$\therefore \frac{dy}{dx} = -\frac{\sqrt{a^{\sin^{-1}t}}}{\sqrt{a^{\cos^{-1}t}}}\left(\frac{a^{\cos^{-1}t}}{a^{\sin^{-1}t}} \times 1\right)$

$= -\frac{a^{\sqrt{\cos^{-1}t}}}{a^{\sqrt{\sin^{-1}t}}}$

$\therefore\ 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{a^{\cos^{-1}t}}{a^{\sin^{-1}t}} = 1 + \frac{y^2}{x^2}$

$= \frac{x^2 + y^2}{x^2}$

21 $\because y = \sec^{-1}\left[\frac{\sqrt{x}+1}{\sqrt{x}-1}\right] + \sin^{-1}\left[\frac{\sqrt{x}-1}{\sqrt{x}+1}\right]$

$= \cos^{-1}\left[\frac{\sqrt{x}-1}{\sqrt{x}+1}\right] + \sin^{-1}\left[\frac{\sqrt{x}-1}{\sqrt{x}+1}\right] = \frac{\pi}{2}$

$\Rightarrow \frac{dy}{dx} = 0\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$

22 Let $y = \tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^1}\right)$

$= \tan^{-1}\left[\frac{2\cdot(3x^{3/2})}{1-(3x^{3/2})^2}\right]$

$= 2\tan^{-1}(3x^{3/2})$

$\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$

$\therefore \frac{dy}{dx} = 2\cdot\frac{1}{1+(3x^{3/2})^2}\cdot 3 \times \frac{3}{2}(x)^{1/2}$

$$= \frac{9}{1+9x^3}\cdot\sqrt{x}$$

$$\therefore g(x) = \frac{9}{1+9x^3}$$

23 Given, $y = \sec(\tan^{-1} x)$

Let $\tan^{-1} x = \theta$

$\Rightarrow \quad x = \tan\theta$

$\therefore \quad y = \sec\theta = \sqrt{1+x^2}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}}\cdot 2x$$

At $x = 1$

$$\frac{dy}{dx} = \frac{1}{\sqrt{2}}$$

24 Given, $y = \tan^{-1}\left[\dfrac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}\right]$.

Put $x^2 = \cos 2\theta$

$$\therefore \quad y = \tan^{-1}\left[\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \theta\right)\right]$$

$$= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x^2$$

$$\therefore \frac{dy}{dx} = 0 + \frac{x}{\sqrt{1-x^4}} = \frac{x}{\sqrt{1-x^4}}$$

25 On putting $x = \sin\theta$ and $y = \sin\phi$, we get

Given equation becomes

$\cos\theta + \cos\phi = a(\sin\theta - \sin\phi)$

$$\Rightarrow 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$

$$= a\left\{2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)\right\}$$

$$\Rightarrow \frac{\theta-\phi}{2} = \cot^{-1} a$$

$\Rightarrow \theta - \phi = 2\cot^{-1} a$

$\Rightarrow \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

26 On putting $x = \sin A$ and $\sqrt{x} = \sin B$

$$y = \sin^{-1}(\sin A\sqrt{1-\sin^2 B} + \sin B\sqrt{1-\sin^2 A})$$

$= \sin^{-1}(\sin A\cos B + \sin B\cos A)$

$= \sin^{-1}[\sin(A+B)]$

$= A + B = \sin^{-1} x + \sin^{-1}\sqrt{x}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{x-x^2}}$$

27 $y = \dfrac{a^{\cos^{-1} x}}{1 + a^{\cos^{-1} x}}$, $z = a^{\cos^{-1} x}$

$$\Rightarrow \quad y = \frac{z}{1+z}$$

$$\Rightarrow \frac{dy}{dz} = \frac{(1+z)1 - z(1)}{(1+z)^2}$$

$$= \frac{1}{(1+z)^2}$$

$$= \frac{1}{(1+a^{\cos^{-1} x})^2}$$

28 Since $g(x)$ is the inverse of $f(x)$

$\therefore \quad f(g(x)) = x$

$\Rightarrow \quad f'(g(x))\cdot g'(x) = 1$

$$\Rightarrow \quad g'(x) = \frac{1}{f'(g(x))} = 1 + (g(x))^5$$

$\Rightarrow \quad g''(x) = 5(g(x))^4\cdot g'(x).$

$= 5(g(x))^4(1 + (g(x))^5)$

29 Since, $\dfrac{dx}{dy} = \left(\dfrac{dy}{dx}\right)^{-1}$

$$\Rightarrow \frac{d^2x}{dy^2} = -\left(\frac{dy}{dx}\right)^{-2}\frac{d^2y}{dx^2}\cdot\frac{dx}{dy}$$

$$= -\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$$

30 Given, $y = \tan^{-1}\left(\dfrac{\log(e/x^2)}{\log(ex^2)}\right) + \tan^{-1}\left(\dfrac{3+2\log x}{1-6\log x}\right)$

$$\therefore \quad y = \tan^{-1}\left(\frac{\log e - \log x^2}{\log e + \log x^2}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$$

$$= \tan^{-1}\left(\frac{1-2\log x}{1+2\log x}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$$

$= \tan^{-1}(1) - \tan^{-1}(2\log x) + \tan^{-1}(3) + \tan^{-1}(2\log x)$

$= \tan^{-1}(1) + \tan^{-1}(3)$

Now, $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} = 0$

31 Since, $y = f(x)$ is symmetrical about the Y-axis

$\therefore \quad f(x)$ is an even function.

Also, as $y = g(x)$ is symmetrical about the origin

$\therefore \quad g(x)$ is an odd function.

Thus, $h(x) = f(x)\cdot g(x)$ is an odd function.

or $\quad h(x) = -h(-x)$

Now, $h'(x) = h'(-x)$

and $\quad h''(x) = -h''(-x)$

$\Rightarrow \quad h''(0) = -h''(0)$

$\Rightarrow \quad h''(0) = 0$

32 Since, $\begin{vmatrix} f'(x) & f(x) \\ f''(x) & f'(x) \end{vmatrix} = 0$

$\therefore \quad (f'(x))^2 - f''(x)\cdot f(x) = 0$

$$\Rightarrow \quad \frac{(f'(x))^2 - f''(x)\cdot f(x)}{(f'(x))^2} = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left[\frac{f(x)}{f'(x)}\right] = 0$$

$$\Rightarrow \quad \frac{f(x)}{f'(x)} = c, \text{(constant)}$$

On putting $x = 0$, we get

$$\frac{1}{2} = c$$

$$\Rightarrow \quad \frac{f(x)}{f'(x)} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)} = 2$$

$$\Rightarrow \quad \frac{d}{dx}(\log f(x)) = 2$$

$\Rightarrow \quad \log(f(x)) = 2x + k$

On putting $x = 0$, we get $0 = k$

$\Rightarrow \quad \log(f(x)) = 2x$

$\Rightarrow \quad f(x) = e^{2x}$

Now, $\displaystyle\lim_{x\to 0}\frac{f(x)-1}{x} = \lim_{x\to 0}\frac{e^{2x}-1}{2x}\cdot 2 = 2.$

33 $f''(x)$

$$= \begin{vmatrix} \frac{d^2}{dx^2}(3x^2) & \frac{d^2}{dx^2}(\cos x) & \frac{d^2}{dx^2}(\sin x) \\ 6 & -1 & 0 \\ P & P^2 & P^3 \end{vmatrix}$$

$$= \begin{vmatrix} 6 & -\cos x & -\sin x \\ 6 & -1 & 0 \\ P & P^2 & P^3 \end{vmatrix}$$

$$\therefore \quad f''(0) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ P & P^2 & P^3 \end{vmatrix} = 0, \text{ which is}$$

independent of P.

34 I. Let $y = (\log x)^{\log x}$

On taking log both sides, we get

$\log y = \log(\log x)\log x$

$\Rightarrow \quad \log y = \log x \log[\log x]$

$[\because \log m^n = n\log m]$

On differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = (\log x)\frac{d}{dx}\{\log(\log x)\} + \log(\log x)\frac{d}{dx}\log x$$

$$= (\log x)\frac{1}{\log x}\frac{1}{x} + \log(\log x)\frac{1}{x}$$

$$= \frac{1}{x}\{1 + \log(\log x)\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}\{1 + \log(\log x)\}$$
$$= \frac{(\log x)^{\log x}}{x}\{1 + \log(\log x)\}$$
$$= (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

II. Let $y = \cos(a\cos x + b\sin x)$.
On differentiating w.r.t. x, we get
$$\frac{d}{dx}\{\cos(a\cos x + b\sin x\}$$
$$= -\sin(a\cos x + b\sin x)\frac{d}{dx}(a\cos x + b\sin x)$$
$$= -\sin(a\cos x + b\sin x)[-a\sin x + b\cos x]$$
$$= (a\sin x - b\cos x)\sin(a\cos x + b\sin x)$$

35 Given, $u = f(\tan x)$
$$\Rightarrow \frac{du}{dx} = f'(\tan x)\sec^2 x$$
and $v = g(\sec x)$
$$\Rightarrow \frac{dv}{dx} = g'(\sec x)\sec x\tan x$$
$$\therefore \frac{du}{dv} = \frac{(du/dx)}{(dv/dx)} = \frac{f'(\tan x)}{g'(\sec x)}\cdot\frac{1}{\sin x}$$
$$\therefore \left(\frac{du}{dv}\right)_{x=\pi/4} = \frac{f'(1)}{g'(\sqrt{2})}\cdot\sqrt{2}$$
$$= \frac{2}{4}\cdot\sqrt{2} = \frac{1}{\sqrt{2}}$$

SESSION 2

1 We have, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x)), x \in R$

Note that, for $x \to 0, \log 2 > \sin x$
$$\therefore f(x) = \log 2 - \sin x$$
$$\Rightarrow g(x) = \log 2 - \sin(f(x))$$
$$= \log 2 - \sin(\log 2 - \sin x)$$
Clearly, $g(x)$ is differentiable at $x = 0$ as $\sin x$ is differentiable.

Now,
$$g'(x) = -\cos(\log 2 - \sin x)(-\cos x)$$
$$= \cos x.\cos(\log 2 - \sin x)$$
$$\Rightarrow g'(0) = 1\cdot\cos(\log 2)$$

2 We have,
$$y = \sin x\cdot\sin 2x\cdot\sin 3x\cdot\ldots\sin nx$$
$$\therefore y' = \cos x\cdot\sin 2x\cdot\sin 3x\ldots\sin nx$$
$$+ \sin x\cdot(2\cos 2x)\sin 3x\ldots\sin nx$$
$$+ \sin x\cdot\sin 2x(3\cos 3x)\ldots\sin nx$$
$$+\ldots+ \sin x\sin 2x\sin 3x\ldots(\cos nx)$$
(by product rule)
$$\Rightarrow y' = \cot x\cdot y + 2\cdot\cot 2x\cdot y + 3\cdot\cot 3x\cdot y + \ldots + n\cdot\cot nx\cdot y$$
$$\Rightarrow y' = y[\cot x + 2\cot 2x + 3\cot 3x + \ldots + n\cot nx]$$
$$\Rightarrow y' = y\cdot\sum_{k=1}^{n} k\cot kx$$

3 $3f(x) - 2f(1/x) = x$ ...(i)

Let $1/x = y$, then
$$3f(1/y) - 2f(y) = 1/y$$
$$\Rightarrow -2f(y) + 3f(1/y) = 1/y$$
$$\Rightarrow -2f(x) + 3f(1/x) = 1/x \quad \ldots(ii)$$
On multiplying Eq. (i) by 3 and Eq. (ii) by 2 and adding, we get
$$5f(x) = 3x + \frac{2}{x}$$
$$\Rightarrow f(x) = \frac{1}{5}\left(3x + \frac{2}{x}\right)$$
$$\Rightarrow f'(x) = \frac{1}{5}\left(3 - \frac{2}{x^2}\right)$$
$$\Rightarrow f'(2) = \frac{1}{5}\left(3 - \frac{2}{4}\right) = \frac{1}{2}$$

4 $f(x) = (\cos x + i\sin x)$
$$(\cos 2x + i\sin 2x)(\cos 3x + i\sin 3x)\ldots(\cos nx + i\sin nx)$$
$$= \cos(x + 2x + 3x + \ldots + nx) + i\sin(x + 2x + 3x + \ldots + nx)$$
$$= \cos\frac{n(n+1)}{2}x + i\sin\frac{n(n+1)}{2}x$$
$$\Rightarrow f'(x) = \left[\frac{n(n+1)}{2}\right]\left[-\sin\frac{n(n+1)}{2}x + i\cos\frac{n(n+1)}{2}x\right]$$
$$f''(x) = -\left[\frac{n(n+1)}{2}\right]^2\left[\cos\frac{n(n+1)}{2}x + i\sin\frac{n(n+1)}{2}x\right]$$
$$= -\left[\frac{n(n+1)}{2}\right]^2\cdot f(x)$$
$$\therefore f''(1) = -\left[\frac{n(n+1)}{2}\right]^2 f(1)$$
$$= -\left[\frac{n(n+1)}{2}\right]^2$$

5 When $x = 0, y > 0 \Rightarrow y = ae^{\pi/2}$

On taking log both sides of the given equation, we get
$$\frac{1}{2}\log(x^2 + y^2) = \log a + \tan^{-1}\left(\frac{y}{x}\right)$$
On differentiating both sides w.r.t. x, we get
$$\frac{1}{2}\times\frac{2x + 2yy'}{x^2 + y^2} = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\times\frac{xy' - y}{x^2}$$
$$\Rightarrow x + yy' = xy' - y \quad \ldots(i)$$
Again, on differentiating both sides w.r.t. x, we get
$$1 + (y')^2 + yy'' = xy'' + y' - y'$$
$$\Rightarrow 1 + (y')^2 = (x - y)y''$$
$$\Rightarrow y'' = \frac{1 + (y')^2}{x - y}$$
When $x = 0$, we get from Eq. (i),
$$y' = -1$$
$$\Rightarrow y''(0) = \frac{2}{-ae^{\pi/2}} = \frac{-2}{a}e^{-\pi/2}$$

6 Given, $y = |\sin x|^{|x|}$

In the neighbourhood of $-\frac{\pi}{6}$, $|x|$ and $|\sin x|$ both are negative

i.e. $y = (-\sin x)^{(-x)}$

On taking log both sides, we get
$$\log y = (-x)\cdot\log(-\sin x)$$
On differentiating both sides, we get
$$\frac{1}{y}\cdot\frac{dy}{dx} = (-x)\left(\frac{1}{-\sin x}\right)\cdot(-\cos x) + \log(-\sin x)\cdot(-1)$$
$$= -x\cot x - \log(-\sin x)$$
$$= -[x\cot x + \log(-\sin x)]$$
$$\Rightarrow \frac{dy}{dx} = -y[x\cot x + \log(-\sin x)]$$
$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } x = -\frac{\pi}{6}} = \frac{(2)^{-\frac{\pi}{6}}}{6}[6\log 2 - \sqrt{3}\pi]$$

7 Since, $f'(x) > g'(x)$
$$\Rightarrow \left(\frac{1}{2}\right)5^{2x+1}\log_e 5 \times 2 > 5^x\log_e 5 + 4\log_e 5$$
$$\Rightarrow 5^{2x}\cdot 5 > 5^x + 4$$
$$\Rightarrow 5\cdot 5^{2x} - 5^x - 4 > 0$$
$$\Rightarrow (5^x - 1)(5\cdot 5^x + 4) > 0$$
$$\therefore 5^x > 1$$
$$\Rightarrow x > 0$$

8 Given, $\frac{d}{dx}\{f'(x)\} = -f(x)$
$$\Rightarrow g'(x) = -f(x)$$
$$[\because g(x) = f'(x), \text{ given}]$$
Also, given $F(x)$
$$= \left\{f\left(\frac{x}{2}\right)\right\}^2 + \left\{g\left(\frac{x}{2}\right)\right\}^2$$
$$\Rightarrow F'(x) = 2\left\{f\left(\frac{x}{2}\right)\right\}f'\left(\frac{x}{2}\right)\cdot\frac{1}{2} + 2\left\{g\left(\frac{x}{2}\right)\right\}\cdot g'\left(\frac{x}{2}\right)\cdot\frac{1}{2} = 0$$
Hence, $f(x)$ is constant. Therefore, $F(10) = 5$.

9 Let $y = f(x)$, then $x = f^{-1}(y)$.

Now, $\frac{d^2x}{dy^2} = (f^{-1})''(y)$
$$\because \frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}$$
$$\therefore \frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dy}{dx}\right)^{-1}$$
$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)^{-1}\cdot\frac{dx}{dy}$$

$$= -\left(\frac{dy}{dx}\right)^{-2} \cdot \frac{d^2y}{dx^2} \cdot \frac{dx}{dy}$$

$$= \frac{\frac{-d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$$

Since, $y = 4$ when $x = 2$

$$\therefore (f^{-1})''(4) = -\frac{\left.\frac{d^2y}{dx^2}\right|_{x=2}}{\left.\left(\frac{dy}{dx}\right)^3\right|_{x=2}} = \frac{-1}{27}$$

10 $f(x) = \sin(\sin x)$

$\Rightarrow f'(x) = \cos x \cdot \cos(\sin x)$

$\Rightarrow f''(x) = -\sin x \cdot \cos(\sin x) - \cos^2 x \cdot \sin(\sin x)$

Now, $g(x) = -[f''(x) + f'(x) \cdot \tan x]$

$= \sin x \cdot \cos(\sin x) + \cos^2 x \cdot \sin(\sin x) - \tan x \cdot \cos x \cdot \cos(\sin x)$

$= \sin x \cdot \cos(\sin x) + \cos^2 x \cdot \sin(\sin x) - \sin x \cdot \cos(\sin x)$

$= \cos^2 x \cdot \sin(\sin x)$

11 We have,

$$\frac{dx}{dt} = a\left[-\sin t\sqrt{\cos 2t} - \frac{\cos t \cdot \sin 2t}{\sqrt{\cos 2t}}\right]$$

$$= \frac{-a\sin 3t}{\sqrt{\cos 2t}}$$

and $\frac{dy}{dt} = a\left[\cos t\sqrt{\cos 2t} - \frac{\sin t \cdot \sin 2t}{\sqrt{\cos 2t}}\right]$

$$= \frac{a\cos 3t}{\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\cot 3t$$

$$\Rightarrow \frac{d^2y}{dx^2} = 3\operatorname{cosec}^2 3t \cdot \frac{dt}{dx}$$

$$= \frac{-3\operatorname{cosec}^2 3t \cdot \sqrt{\cos 2t}}{a\sin 3t}$$

$$= -\left(\frac{3}{a}\right)\operatorname{cosec}^3 3t \cdot \sqrt{\cos 2t}$$

$$\therefore \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2}$$

$$= (1 + \cot^2 3t)^{3/2} \Big/ \left(\frac{-3}{a}\right)\operatorname{cosec}^3 3t\sqrt{\cos 2t}$$

$$\Rightarrow \left|\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \Big/ \frac{d^2y}{dx^2}\right| \text{ at } t = \frac{\pi}{6} \text{ is}$$

$$\frac{a}{3\sqrt{\cos\frac{\pi}{3}}} = \frac{\sqrt{2}a}{3}$$

12 Clearly, $f(x) = e^{g(x)}$

Now, as $g(x+1) = x + g(x)$

$\therefore \quad e^{g(x+1)} = e^{x+g(x)} = e^x \cdot e^{g(x)}$

$\Rightarrow \quad f(x+1) = e^x f(x)$

On taking log both sides, we get

$\ln f(x+1) = \ln(e^x \cdot f(x))$

$$\Rightarrow \frac{1}{f(x+1)} . f'(x+1)$$

$$= 1 + \frac{1}{f(x)} . f'(x)$$

$$\Rightarrow \frac{f'(x+1)}{f(x+1)} - \frac{f'(x)}{f(x)} = 1$$

$$\Rightarrow \frac{f'\left(1+\frac{1}{3}\right)}{f\left(1+\frac{1}{3}\right)} - \frac{f'\left(\frac{1}{3}\right)}{f\left(\frac{1}{3}\right)} = 1$$

$$\frac{f'\left(2+\frac{1}{3}\right)}{f\left(2+\frac{1}{3}\right)} - \frac{f'\left(1+\frac{1}{3}\right)}{f\left(1+\frac{1}{3}\right)} = 1$$

$$\frac{f'\left(n+\frac{1}{3}\right)}{f\left(n+\frac{1}{3}\right)} - \frac{f'\left((n-1)+\frac{1}{3}\right)}{f\left((n-1)+\frac{1}{3}\right)} = 1$$

on adding columnwise, we get

$$\frac{f'\left(n+\frac{1}{3}\right)}{f\left(n+\frac{1}{3}\right)} - \frac{f'\left(\frac{1}{3}\right)}{f\left(\frac{1}{3}\right)} = n$$

13 Since, $g(x) = f^{-1}(x)$

$\therefore \quad f(g(x)) = x \Rightarrow f'(g(x)) \cdot g'(x) = 1$

$$\Rightarrow \quad g'(x) = \frac{1}{f'(g(x))}$$

$$\Rightarrow \quad g'\left(\frac{-7}{6}\right) = \frac{1}{f'\left(g\left(-\frac{7}{6}\right)\right)}$$

$$= \frac{1}{f'\left(f^{-1}\left(\frac{-7}{6}\right)\right)}$$

$$\left[\because f(1) = -4 + 1 + 1 + \frac{1}{2} + \frac{1}{3} = -\frac{7}{6} \quad \therefore f^{-1}\left(\frac{7}{6}\right) = 1\right]$$

$$= \frac{1}{5}$$

$$\left[\because f'(x) = -4e^{\frac{1-x}{2}}\left(-\frac{1}{2}\right) + 1 + x + x^2\right]$$

14 We have, $f(x) = \prod_{i=1}^{100}(x-i)^{i(101-i)}$

$$\Rightarrow \log f(x) = \sum_{i=1}^{100} i(101-i)\log(x-i)$$

$$\frac{1}{f(x)} \cdot f'(x) = \sum_{i=1}^{100} i(101-i) \cdot \frac{1}{x-i}$$

$$\Rightarrow \frac{f'(101)}{f(101)} = \sum_{i=1}^{100} i\frac{(101-i)}{(101-i)}$$

$$= \sum_{i=1}^{100} i = \frac{100(101)}{2} = 5050$$

15 Given, $x = 2t - |t|$ and $y = t^3 + t^2\,|t|$

Clearly, $x = t$, $y = 2t^3$ when $t \geq 0$

and $x = 3t$, $y = 0$ when $t < 0$

On eliminating the parameter t, we get

$$y = \begin{cases} 2x^3, & \text{when } x \geq 0 \\ 0, & \text{when } x < 0 \end{cases}$$

Now, $\frac{dy}{dx} = \begin{cases} 6x^2, & \text{when } x > 0 \\ 0, & \text{when } x < 0 \end{cases}$

$\because \quad (\text{LHD})_{\text{at } x=0} = (\text{RHD})_{\text{at } x=0} = 0$

$\therefore$ Its derivative at $x = 0$

(i.e. at $t = 0$ is 0)